Dynatomic polynomials associated with distinguished polynomials.

SYLLA Djeidi

Université des Sciences, des Techniques et des Technologies de Bamako
Faculté des Sciences et Techniques

VIII-th International Conference on p-adic Mathematical Physics and its applications

Outline

Definitions and reminders

Definitions and reminders

Let K be a field $h \in K[z]$ be unitary polynomial of degree n.
By analogy with the definition of cyclotomic polynomials, for any integer $\nu \geq 1$, one sets $\Phi_{\nu,h}(z) = \prod_{d|\nu} \left(h^{\circ d}(z) - z\right)^{\mu\left(\frac{\nu}{d}\right)}$, where μ is the arithmetical Möbius function, notice that $\Phi_{1,h}(z) = h(z) - z$. At first glance $\Phi_{\nu,h}(z)$ is a rational fraction. But in fact, one can prove that $\Phi_{\nu,h}(z)$ is a polynomial with degree $\deg(\Phi_{\nu,h}) = \sum_{d|\nu} \mu\left(\frac{\nu}{d}\right) n^d$.

Définition

Let ν be a positive integer and h be a polynomial with coefficients in K. The rational fraction $\Phi_{\nu,h}(z) \in K(z)$ is a polynomial said to be dynatomic polynomial associated with polynomial h.

The degree of $\Phi_{\nu,\ h}$ is $\deg\left(\Phi_{\nu,\ h}\right) = \sum_{d|\nu} \mu\left(\frac{\nu}{d}\right)(\deg h)^d$.

Definitions and reminders

Définition

An element α of the algebraic closure $\tilde{K}$ of K is a periodic point of h, if there exists an integer $\nu \geq 1$ such that $h^{\circ\nu}(\alpha) = \alpha$. Hence α is said to be a periodic point of order ν or a ν-periodic point.
Furthermore α is said to be a periodic primitive point of ν or primitive ν-periodic point if it is ν-periodic and for any integer $1 \leq j < \nu$, one has $h^{\circ j}(\alpha) \neq \alpha$.

Lemma

If the polynomial h admits a primitive ν-periodic point α, then $\Phi_{\nu,h}(\alpha) = 0$.
Conversely, if one supposes that the ν-dynatomic polynomial $\Phi_{\nu,h}$ is separable, i.e. its roots are simple, then the primitive ν-periodic points of h are the roots of $\Phi_{\nu,h}$.

The polynomials h written in the form $h(z) = z + g(z)$

Outline

1. Definitions and reminders.
2. The polynomials h written in the form $h(z) = z + g(z)$
   - The polynomials $g_\nu(z)$.
   - Examples in a complete valued field with residue characteristic $p \neq 0$.
   - The p^ℓ-dynatomic polynomial for $h(z) = a_0 + z + z^q$, $0 \neq a_0 \in p\mathbb{Z}_p$.
3. The p-adic 3-periodic points of $h(z) = a_0 + z + z^q$, $v_p(a_0) \geq 1$
   - The case $p = 2$ and $q = 2$
   - The case $p = 3$ and $q = 3$
   - The case $p = 5$ and $q = 5$
   - The case $p = 7$ and $q = 7$

The polynomials h written in the form $h(z) = z + g(z)$.

I. The polynomials $g_\nu(z)$.

- Let us consider the polynomial h written in the form $h(z) = z + g(z)$, $\deg(h) \geq 1$.
  It follows that if $h(z) \neq z$, one has $\deg(h) = \deg(g)$.

- For any positive integer ν, we shall put $h^{\circ\nu} = z + g_\nu(z)$, with $g_0(z) = 0$ and $g_1(z) = g(z)$. One has:
  $g_\nu(z) = h^{\circ\nu}(z) - z = \prod_{d|\nu} \Phi_{d,\ h}(z)$ and $\Phi_{\nu,\ h}(z) = \prod_{d|\nu} g_d(z)^{\mu(\frac{\nu}{d})}$.

Outline

1. Definitions and reminders.
2. The polynomials h written in the form $h(z) = z + g(z)$
   - The polynomials $g_\nu(z)$.
   - Examples in a complete valued field with residue characteristic $p \neq 0$.
   - The p^ℓ-dynatomic polynomial for $h(z) = a_0 + z + z^q$, $0 \neq a_0 \in p\mathbb{Z}_p$.
3. The p-adic 3-periodic points of $h(z) = a_0 + z + z^q$, $v_p(a_0) \geq 1$
   - The case $p = 2$ and $q = 2$
   - The case $p = 3$ and $q = 3$
   - The case $p = 5$ and $q = 5$
   - The case $p = 7$ and $q = 7$

The polynomials h written in the form $h(z) = z + g(z)$.

II. Examples in a complete valued field with residue characteristic $p \neq 0$

- Let us remind that if A is a commutative local ring with maximal ideal $\mathcal{M}$, a unitary polynomial $g(z) = a_0 + a_1 z + \cdots + a_{s-1} z^{s-1} + z^s \in A[z]$ is distinguished if a_j belongs to $\mathcal{M}$, for $0 \leq j \leq s-1$.
- Let us notice that if A is a discrete valuation ring, any Eisenstein polynomial is distinguished.

The polynomials h written in the form $h(z) = z + g(z)$.

Let L be a complete ultrametric valued field, we denote by Λ_L its ring of valuation, by $\mathcal{M}_L$ the maximal ideal of Λ_L and $\overline{L} = \Lambda_L/\mathcal{M}_L$, the residue field of L. In the sequel we assume that the residue characteristic of L is a prime number p.

Theorem

Let L be a complete ultrametric valued field of residue characteristic $p \neq 0$, q a power of p. Set $h(z) = z + g(z)$.
Reducing the polynomial $g_\nu(z) = h^{\circ\nu}(z) - z$, $\nu \geq 1$, modulo the ideal $\mathcal{M}_L[z]$ of $\Lambda_L[z]$, one obtains in $\overline{L}[z]$ the formula: $\overline{g}_\nu(z) = \sum_{j=0}^{\nu} \binom{\nu}{j} z^{q^j}$

The polynomials h written in the form $h(z) = z + g(z)$.

Corollary

Let L be a complete ultrametric valued field of residue characteristic $p \neq 0$.
Let $g(z) = a_0 + a_1 z + \cdots + a_{q-1} z^{q-1} + z^q$ be a distinguished polynomial over the valuation ring Λ_L of L with degree q a power of p. Then:

- For any integer $\nu \geq 1$, the polynomial $\overline{g}_\nu(z) \in \overline{L}[z]$ is an additive polynomial.
- Therefore $\overline{g}(0) = 0$, i.e. $|g_\nu(0)| < 1$.

Corollary

Let L be a complete ultrametric valued field of residue characteristic $p \neq 0$. Let $g(z) = a_0 + a_1 z + \cdots + a_{q-1} z^{q-1} + z^q$ be a distinguished polynomial over the valuation ring Λ_L of L with degree q a power of p. Then for any integer $\nu \geq 1$, the derivative of the polynomial $h^{\circ\nu}(z) = z + g_\nu(z)$ is such that $(h^{\circ\nu})'(z) \equiv 1(\text{mod. } \mathcal{M}_L[z])$, where $\mathcal{M}_L$ is the maximal ideal of Λ_L.

The polynomials h written in the form $h(z) = z + g(z)$.

Lemma

Let L be a complete ultrametric value field with valuation ring Λ.
Let $f(z) = c_0 + c_1 z + \cdots + c_{m-1} z^{m-1} + c_m z^m \in \Lambda_L[z]$, such that $|c_m| = 1$.
If α is a root of f in an algebraic closure of L, then $|\alpha| \leq 1$.

Proposition

Let L be a complete ultrametric valued field of residue characteristic $p \neq 0$.
Let $g(z) = a_0 + a_1 z + \cdots + a_{q-1} z^{q-1} + z^q \in \Lambda_L[z]$ be a distinguished polynomial of degree q a power of p. Put $h(z) = z + g(z)$. Then:

- For any integer $\nu \geq 1$, the ν-periodic points of h are indifferent points.
- Let α be a periodic point of order ν in an algebraic closure of L. One has $|\alpha| \leq 1$. Furthermore in the field extension $K = L[\alpha]$ of L, the closed disc $D_K^+(\alpha, 1)$ is a Siegel disc of α in K.

Outline

1. Definitions and reminders.
2. The polynomials h written in the form $h(z) = z + g(z)$
   - The polynomials $g_\nu(z)$.
   - Examples in a complete valued field with residue characteristic $p \neq 0$.
   - The p^ℓ-dynatomic polynomial for $h(z) = a_0 + z + z^q$, $0 \neq a_0 \in p\mathbb{Z}_p$.
3. The p-adic 3-periodic points of $h(z) = a_0 + z + z^q$, $v_p(a_0) \geq 1$
   - The case $p = 2$ and $q = 2$
   - The case $p = 3$ and $q = 3$
   - The case $p = 5$ and $q = 5$
   - The case $p = 7$ and $q = 7$

The polynomials h written in the form $h(z) = z + g(z)$.

III.The p^ℓ-dynatomic polynomial for $h(z) = a_0 + z + z^q$, $0 \neq a_0 \in p\mathbb{Z}_p$.

Let $q = p^s$ be a power of the prime p, The polynomial $g(z) = z^q$ is a particular distinguished polynomial with coefficients in the ring of p-adic integer $\mathbb{Z}_p$.

Proposition (Moton and Patel, 1994)

Let $q = p^s$ be a power of the prime number p.
Assume that $g(z) = z^q$.
Then for any power p^ℓ of p the dynatomic polynomial $\Phi_{p^\ell, z+z^q}$ is an Eisenstein polynomial with coefficients in the ring of p-adic integers $\mathbb{Z}_p$ and degree $q^{p^\ell} - q^{p^{\ell-1}}$.

The polynomials h written in the form $h(z) = z + g(z)$.

Corollary

Let $h(z) = z + z^q$ where $q > 1$ is a power of the prime number p.

- For any power p^ℓ, $\ell \geq 1$ of p; the primitive p^ℓ-periodic points α of $h(z) = z + z^q$ are the roots of the dynatomic polynomial $\Phi_{p^\ell, z+z^q}$.
- Each algebraic extension $K = \mathbb{Q}_p[\alpha]$ is totally ramified of degree $q^{p^\ell} - q^{p^{\ell-1}}$ and contains the p^ℓ-cycle $\{\alpha, h(\alpha), \cdots, h^{p^{\ell-1}}(\alpha)\}$.

We shall give below a kind of generalization of the above results obtained for the polynomial $h(z) = z + z^q$.

The polynomials h written in the form $h(z) = z + g(z)$.

Theorem (Diarra and Sylla, 2020)

Let p be a prime number.
Let q be a power of p such that $q \geq 3$. Consider the distinguished polynpomial of the form $g(z) = a_0 + z^q \in \mathbb{Z}_p[z]$, with $a_0 \neq 0$ and $h(z) = a_0 + z + z^q$.

- For $1 \leq \ell \leq q - 2$, the dynatomic polynomial $\Phi_{p^\ell,h}(z)$ is an Eisenstein polynomial. The primitive p^ℓ-periodic points of h are the roots of $\Phi_{p^\ell,h}(z)$.
- If α is a primitive p^ℓ-periodic points, then the field extension $K = \mathbb{Q}_p[\alpha]$ of $\mathbb{Q}_p$ is totally ramified of degree $q^{p^\ell} - q^{p^{\ell-1}}$.
- Furthermore, if β is a p^ℓ-periodic point of h, then or β is a fixed point of h: hence a root of $g(z) = a_0 + z^q$, or there exists an integer j with $1 \leq j \leq \ell$ such β is a primitive p^j-periodic point of h.

The polynomials h written in the form $h(z) = z + g(z)$.

Corollary

With the above notations, if α is a primitive p^ℓ-periodic, the totally ramified extension $K = \mathbb{Q}_p[\alpha]$ of $\mathbb{Q}_p$ contains the p^ℓ-cycle $\{\alpha, h(\alpha), \cdots, h^{\circ p^{\ell-1}}(\alpha)\}$. The number of p^ℓ-cycles in the Galois field of the polynomial $\Phi_{p^\ell,h}(z)$ is equal to $\frac{1}{p^\ell}\left(q^{p^\ell} - q^{p^{\ell-1}}\right)$.

Remark

The results stated in this subsection remain true if in place of $\mathbb{Z}_p$, one takes $a_0 \in \mathcal{M}_L$, the maximal ideal of Λ_L the valuation ring of a finite unramified extension L of $\mathbb{Q}_p$; because one has $\mathcal{M}_L = p\Lambda_L$.

The p-adic 3-periodic points of $h(z) = a_0 + z + z^q$, $v_p(a_0) \geq 1$

The p-adic 3-periodic points of $h(z) = a_0 + z + z^q$, $v_p(a_0) \geq 1$.

Let q be a power of p. Consider the distinguished polynomial of the form $g(z) = a_0 + z^q \in \Lambda_L[z]$, with coefficients in the valuation ring Λ_L of a complete ultrametric valued field L residue characteristic p, $a_0 \neq 0$ and $h(z) = a_0 + z + z^q$.
If ν is prime number the ν-dynatomic polynomial associate to $h(z)$ is give by $\Phi_{\nu,h}(z) = \dfrac{h^{\circ\nu}(z) - z}{h(z) - z} = \dfrac{g_\nu(z)}{g(z)}$. Therefore $\overline{\Phi}_{\nu,h}(z) = \sum_{j=1}^{\nu} \binom{\nu}{j} z^{q^j - q} = \left(\sum_{j=1}^{\nu} \binom{\nu}{j} z^{q^{j-1} - 1} \right)^q$. In particular, for $\nu = 3$, one has $\overline{\Phi}_{3,\ h}(z) = (3 + 3z^{q-1} + z^{q^2-1})^q$.

Corollary

Let us notice that if $p \neq 3$, then any root α of $\Phi_{3,h}$ is of absolute value $|\alpha| = 1$.

The p-adic 3-periodic points of $h(z) = a_0 + z + z^q$, $v_p(a_0) \geq 1$.

Proposition

Let L be a complete ultrametric valued field of residue characteristic $p \neq 0$ and $g(z) = a_0 + z^q$ be a binomial that is a distinguished polynomial in the ring of valuation Λ_L of L, with q a power of p. Then the set of the primitive 3-periodic points of $h(z) = a_0 + z + z^q$ is equal to the set of roots of the 3-dynatomic polynomial $\Phi_{3,h}$.

A consequence is that the polynomial $\Phi_{3,h}$ is separable.

Corollary

The number of 3-cycles of $h(z) = a_0 + z + z^q$ is equal to $\dfrac{1}{3}\deg(\Phi_{3,\ h}) = \frac{1}{3}(q^3 - q)$.

Outline

The p-adic 3-periodic points of $h(z) = a_0 + z + z^q$, $v_p(a_0) \geq 1$.

I. The case $p = 2$ and $q = 2$.

We take here $L = \mathbb{Q}_2$, $q = 2$. Then $g(z) = a_0 + z^2$, with $a_0 \in 2\mathbb{Z}_2$ and $h(z) = z + g(z) = a_0 + z + z^2$.
Reducing modulo $2\mathbb{Z}_2[z]$, one obtains directly the formula $\overline{\Phi}_{3,\ h}(z) = z^6 + z^2 + 1 = (z^3 + z + 1)^2 \in \mathbb{F}_2[z]$.

Lemma

Assume that a_0 belongs to $4\mathbb{Z}_2$.
Then the 3-dynatomic polynomial $\Phi_{3,h}$ of the polynomial $h(z) = a_0 + z + z^2$ is irreducible.

Remark

Assume that $a_0 = 2b_0$, with $|b_0| = 1$.
Unfortunately, we cannot apply Schönemann criterium here to $\Phi_{3,h}$.

The p-adic 3-periodic points of $h(z) = a_0 + z + z^q$, $v_p(a_0) \geq 1$.

Theorem (Diarra and Sylla 2020)

Assume that a_0 belongs to $4\mathbb{Z}_2$.
Let $h(z) = a_0 + z + z^2$ and β be a root of the irreducible polynomial $\Phi_{3,h}$.
Put $K = \mathbb{Q}_2[\beta]$. Let $\omega = \lim_{n \to +\infty} \beta^{8^n} \in K$ and $E = \mathbb{Q}_2[\omega]$. Then

- E is the maximal unramified extension of $\mathbb{Q}_2$ contained in K.
- The algebraic extension $K|E$ is totally ramified of degree 2 with an uniformizer $\pi = \beta^3 + \beta + 1$.
  Moreover $K|\mathbb{Q}_2$ is a Galois extension, its Galois group is isomorphic to $\mathbb{Z}/6\mathbb{Z}$.
- The number of the 3-cycles of the polynomial $h(z) = a_0 + z + z^2$ is 2.
  If γ is the conjugate of β in the quadratic extension $K|E$, then the 3-cycles of h are $(\beta,\ h(\beta),\ h^{\circ 2}(\beta))$ and $(\gamma,\ h(\gamma),\ h^{\circ 2}(\gamma))$.

Outline

1. Definitions and reminders.
2. The polynomials h written in the form $h(z) = z + g(z)$
   - The polynomials $g_\nu(z)$.
   - Examples in a complete valued field with residue characteristic $p \neq 0$.
   - The p^ℓ-dynatomic polynomial for $h(z) = a_0 + z + z^q$, $0 \neq a_0 \in p\mathbb{Z}_p$.
3. The p-adic 3-periodic points of $h(z) = a_0 + z + z^q$, $v_p(a_0) \geq 1$
   - The case $p = 2$ and $q = 2$
   - The case $p = 3$ and $q = 3$
   - The case $p = 5$ and $q = 5$
   - The case $p = 7$ and $q = 7$

The p-adic 3-periodic points of $h(z) = a_0 + z + z^q$, $v_p(a_0) \geq 1$.

II. The case $p = 3$ and $q = 3$.

If $p = 3$, and $g(z) = a_0 + z^3 \in \mathbb{Q}_3[z]$ is a distinguished polynomial and $h(z) = a_0 + z + z^3$, one has $\overline{\Phi}_{3,h}(z) = 3 + 3z^{3^2-3} + z^{3^3-3} = z^{24}$.

Corollary

If $p = 3$ and $g(z) = a_0 + z^3 \in \mathbb{Q}_3[z]$ is a distinguished polynomial, then for $h(z) = a_0 + z + z^3$, the 3-dynatomic polynomial $\Phi_{3,h}$ is an Eisenstein polynomial.

The p-adic 3-periodic points of $h(z) = a_0 + z + z^q$, $v_p(a_0) \geq 1$.

Proposition (Diarra and Sylla, 2020)

Let $\mathbb{K}_3$ be the subfield of the 3-adic complex number field $\mathbb{C}_3$ generated by the roots of the 3-dynatomic polynomial $\Phi_{3,a_0+z+z^3}(z)$, $|a_0| < 1$. Then

- The number of 3-orbits of h is equal $\dfrac{24}{3} = 8$.
- For any primitive 3-periodic point of $h(z) = a_0 + z + z^3$ the orbit $\{\beta, h(\beta), h^{\circ 2}(\beta)\}$ is contained in $\mathbb{Q}_3[\beta]$. There are at most 8 distinct subfields of $\mathbb{K}_3$ defined by the roots of $\Phi_{3,a_0+z+z^3}(z)$.

Outline

1. Definitions and reminders.

2. The polynomials h written in the form $h(z) = z + g(z)$
   - The polynomials $g_\nu(z)$.
   - Examples in a complete valued field with residue characteristic $p \neq 0$.
   - The p^ℓ-dynatomic polynomial for $h(z) = a_0 + z + z^q$, $0 \neq a_0 \in p\mathbb{Z}_p$.

3. The p-adic 3-periodic points of $h(z) = a_0 + z + z^q$, $v_p(a_0) \geq 1$
   - The case $p = 2$ and $q = 2$
   - The case $p = 3$ and $q = 3$
   - The case $p = 5$ and $q = 5$
   - The case $p = 7$ and $q = 7$

The p-adic 3-periodic points of $h(z) = a_0 + z + z^q$, $v_p(a_0) \geq 1$.

III. The case $p = 5$ and $q = 5$.

Let $g(z) = a_0 + z^5 \in \mathbb{Z}_5[z]$ be a distinguished polynomial and $h(z) = z + g(z) = a_0 + z + z^5$. One has $\overline{\Phi}_{3,h}(z) = (3 + 3z^4 + z^{5^2-1})^5 = (3 + 3z^4 + z^{24})^5$

Proposition

For $a_0 \in 5\mathbb{Z}_5$, the polynomial $\Phi_{3,\ a_0+z+z^5}(z) \in \mathbb{Q}_5[z]$ is irreducible.

The p-adic 3-periodic points of $h(z) = a_0 + z + z^q$, $v_p(a_0) \geq 1$.

Theorem (Diarra and Sylla)

Assume that a_0 belongs to $5\mathbb{Z}_5$.
Let $h(z) = a_0 + z + z^5$ and β be a root of the irreducible polynomial $\Phi_{3,h}$.
Put $K = \mathbb{Q}_5[\beta]$. Let $\omega = \lim_{n \to +\infty} \beta^{5^{24n}} \in K$ and $E = \mathbb{Q}_5[\omega]$. Then

- E is the maximal unramified extension of $\mathbb{Q}_5$ contained in K.
- The algebraic extension $K|E$ is totally ramified of degree 5 with an uniformizer $\pi = 3 + 3\beta^4 + \beta^{24}$.
- The number of 3-cycles of $h(z) = a_0 + z + z^5$ is equal to 40.

Outline

The p-adic 3-periodic points of $h(z) = a_0 + z + z^q$, $v_p(a_0) \geq 1$.

IV. The case $p = 7$ and $q = 7$.

Let $g(z) = a_0 + z^7 \in \mathbb{Z}_7[z]$ be a distinguished polynomial and $h(z) = z + g(z) = a_0 + z + z^7 \in \mathbb{Z}_7[z]$. One has $\overline{\Phi}_{3,h}(z) = (3 + 3z^6 + z^{48})^7$.

$$\deg(\Phi_{3,h}(z)) = 7^3 - 7 = 336 \text{ and } \Phi_{3,h}(z) = \prod_{i=1}^{336} (z - \beta_i) \in \mathbb{C}_7[z].$$

We shall put $\Phi_{3,h}(z) = \prod_{j=1}^{8} \Phi_j(z)$.

The p-adic 3-periodic points of $h(z) = a_0 + z + z^q$, $v_p(a_0) \geq 1$.

For the polynomials $\Phi_j(z)$, $1 \leq j \leq 6$

Let us consider Ψ written in the form $\Psi(z) = \prod_{j=1}^{6} \Phi_j(z)$, one has

$$\overline{\Psi}(z) = \prod_{\ell=1}^{6}(z - \overline{\ell}) = \prod_{\eta \in \mathbb{F}_7^\star} (z - \eta).$$

Let $\mathcal{Z}_\psi$ be the set of roots of Ψ. Since $\overline{h}(z)$ permutes the roots of $\overline{\Psi}(z)$ if $\beta \in \mathcal{Z}_\Psi$, then the 3-cycle $(\beta,\ h(\beta),\ h^{\circ 2}(\beta))$ is in $\mathcal{Z}_\psi$.

Proposition

Let β be a root of Φ_j, $1 \leq j \leq 6$, then:

- $\mathbb{Q}_7[\beta]$ is equal to $\mathbb{Q}_7$ or is a totally ramified extension of $\mathbb{Q}_7$.
- The number of the 3-cycles is 14

The p-adic 3-periodic points of $h(z) = a_0 + z + z^q$, $v_p(a_0) \geq 1$.

For the Polynomial $\Phi_7(z)$

The polynomial Φ_7 is such that $\overline{\Phi}_7(z) = w_7(z)^7 = (z^6 - 3)^7 \in \mathbb{F}_7[z]$.

Proposition

Put $\mathcal{Z}_7 = \{\beta \in \mathbb{C}_7 / w_7(\overline{\beta}) = 0\}$, the set of roots of Φ_7. Then if $\beta \in \mathcal{Z}_7$, one has $h(\beta) \in \mathcal{Z}_7$. Furthermore $\mathcal{Z}_7$ is a partition of 14 subsets of 3-cycles.

The p-adic 3-periodic points of $h(z) = a_0 + z + z^q$, $v_p(a_0) \geq 1$.

For the Polynomial $\Phi_8(z)$

The polynomial Φ_8 is such that
$\overline{\Phi}_8(z) = w_8(z)^7 = (z^{36} + 4z^{30} + 5z^{18} + 2z^{12} + 1)^7$.

Proposition

Let $\mathcal{Z}_8 = \{\beta \in \mathbb{C}_7 / w_8(\overline{\beta}) = 0\}$, the set of roots of $\Phi_8(z)$.
Let us consider $\mathcal{Z}_\phi$, the set of the 3-primitive periodic points, that is the set of roots of 3-dynatomic polynomial $\Phi_{3,\ h}$.
Then $\mathcal{Z}_8 = \mathcal{Z}_\phi \setminus \left(\mathcal{Z}_\psi \cup \mathcal{Z}_7\right)$.
Since $\mathcal{Z}_\psi$ and $\mathcal{Z}_7$ are invariant by h then $\mathcal{Z}_8$ is also invariant by h. Hence for any $\beta \in \mathcal{Z}_8$ the 3-cycle $(\beta,\ h(\beta),\ h^2(\beta))$ is in $\mathcal{Z}_8$. Furthermore $\mathcal{Z}_8$ is a partition of $84 = \dfrac{252}{3}$ subsets of 3-cycles.

Thanks